Reading course on
Elliptic operators, topology and asymptotic methods

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Sommersemester 2026

Updated on May 15, 2026.

In this reading course, given at the University of Münster during the summer semester of 2026, we will be covering parts of [3]. We will be meeting regularly to discuss the material and any questions that arise.

For some concepts, the lectures notes [1] and [2] might come in handy. Note that not everything in those lecture notes is part of this course, but that some concepts are defined in these lecture notes in a way which slightly differs from the book [3]. Thus, these lecture notes can be helpful for getting a broader/alternate point of view of some of the concepts from this course.

Preliminary plan for the course

This plan is tentative and will be adjusted as the course progresses, depending on pace and interests.

- Chapter 1 (recap of basic Riemannian geometry, vector bundles, connections, curvature, and exterior calculus)
- Chapter 3
- Chapter 5
- Chapter 7
- Chapter 8
- Chapter 11
- Additional topics in dimension four (notes/references will be posted here later)

First meeting (April 21, 2026)

Typo in the book

There is a typo in [3, Question 1.28]. The correct formula is

$$d\alpha(X_0,\dots,X_p) = \sum_i (-1)^i X_i(\alpha(X_0,\dots,\widehat{X_i},\dots,X_p)) \\ + \sum_{i<j}(-1)^{i+j}\alpha([X_i,X_j],X_0,\dots,\widehat{X_i},\dots,\widehat{X_j},\dots,X_p).$$

Motivating example for the course

As a motivating example for this course, we consider the Gauss–Bonnet theorem. For a closed Riemannian 2-manifold (M,g), the Gauss–Bonnet theorem tells us that

$$\chi(M) = \frac{1}{2\pi}\int_M K\,dA,$$

where K is the Gaussian curvature and $\chi(M)$ is the Euler characteristic, a topological invariant which can be expressed in terms of spaces of differential forms:

$$\chi(M) = \sum_{k=0}^{2}(-1)^k \dim(\ker(d_k)/\operatorname{im}(d_{k-1})),$$

where $d_k : \Omega^k(M) \to \Omega^{k+1}(M)$ is the exterior derivative on k-forms. There is a natural generalization of the Laplace operator to differential forms, $\Delta_k = d_{k-1}d_{k-1}^* + d_k^* d_k : \Omega^k(M) \to \Omega^k(M)$, and a subject known as Hodge theory tells us that

$$\chi(M) = \sum_{k=0}^{2}(-1)^k \dim(\ker(\Delta_k)).$$

We will now introduce a square root, D, for the Laplacian. Unfortunately, there is no differential operator $D_k : \Omega^k(M) \to \Omega^k(M)$ which is a square root for Δ_k. However, we can consider the space of mixed differential forms $\Omega^*(M) = \bigoplus_{k=0}^2 \Omega^k(M)$, and view the Laplacian as an operator $\Delta : \Omega^*(M) \to \Omega^*(M)$ which acts separately on each degree, without mixing the degrees. If we also consider d and d^* to be operators on $\Omega^*(M)$ and define $D = d + d^*$, then $D^2 = dd^* + d^*d = \Delta$ because $dd = d^*d^* = 0$. Interestingly enough, the operator D does mix the different degrees, but it does so in a controlled fashion: the subspace $\Omega^{\text{even}}(M) = \Omega^0(M) \oplus \Omega^2(M)$ of forms of even degree gets mapped into the subspace $\Omega^{\text{odd}}(M) = \Omega^1(M)$ of forms of odd degree, and vice versa. These subspaces appear naturally in the expression for $\chi(M)$:

$$\chi(M) = \sum_{k=0}^{2}(-1)^k \dim(\ker(\Delta|_{\Omega^k(M)})) = \dim(\ker(\Delta|_{\Omega^{\text{even}}(M)})) - \dim(\ker(\Delta|_{\Omega^{\text{odd}}(M)})).$$

The choice of $D = d + d^*$ as a square root of the Laplacian is not arbitrary. It arises naturally when studying so-called Clifford bundles, of which the bundle of mixed differential forms is an example. Quite generally, such a structure on a vector bundle automatically gives rise to a first-order differential operator whose square is a "Laplace-like" operator (and in fact, an alternative viewpoint is that the choice of $\Delta = dd^* + d^*d$ as a generalization of the Laplacian came from the operator D, rather than the other way around).

Now, how would one go about calculating these dimensions? In fact, how would one go about constructing a harmonic differential form? Given a differential form ω_0, one can formulate the so-called heat equation $\partial_t\omega_t = -\Delta\omega_t$. As for scalar functions on Euclidean space, this equation admits a unique solution ω_t, smooth in both space and time, for all $t \geq 0$, and provides a so-called heat flow for differential forms. It also turns out that this converges to an equilibrium solution ω_∞ as $t \to \infty$. The fact that this is an equilibrium solution means precisely that $\Delta\omega_\infty = 0$, and in fact, ω_∞ is the orthogonal projection of ω_0 onto the space of harmonic forms in an L^2 sense. If the heat flow operator is denoted $H_t : \Omega^*(M) \to \Omega^*(M)$, then intuitively, one might expect that

$$\dim(\ker(\Delta|_{\Omega^{\text{even}}(M)})) = \operatorname{tr}(\operatorname{proj}_{\ker(\Delta|_{\Omega^{\text{even}}(M)})}) = \lim_{t\to\infty} \operatorname{tr}(H_t|_{\Omega^{\text{even}}(M)}),$$

and analogously for the odd forms. This turns out to be true, but requires careful justification (and a proper definition of trace).

In particular,

$$\chi(M) = \lim_{t\to\infty}(\operatorname{tr}(H_t|_{\Omega^{\text{even}}(M)}) - \operatorname{tr}(H_t|_{\Omega^{\text{odd}}(M)})).$$

Somewhat surprisingly, this is not just an equality in the limit, but it holds for all finite $t > 0$ as well:

$$\chi(M) = \operatorname{tr}(H_t|_{\Omega^{\text{even}}(M)}) - \operatorname{tr}(H_t|_{\Omega^{\text{odd}}(M)}).$$

An intuitive argument for this fact is as follows. Introduce the so-called grading operator $\varepsilon : \Omega^*(M) \to \Omega^*(M)$, which acts as the identity on even forms, but flips the sign of odd forms. Then

$$\operatorname{tr}(H_t|_{\Omega^{\text{even}}(M)}) - \operatorname{tr}(H_t|_{\Omega^{\text{odd}}(M)}) = \operatorname{tr}(\varepsilon \circ H_t),$$

and the derivative of this with respect to t is

$$\operatorname{tr}(\varepsilon \circ \partial_t H_t) = -\operatorname{tr}(\varepsilon \circ \Delta \circ H_t) = -\operatorname{tr}(\varepsilon \circ D \circ D \circ H_t).$$

Since D flips the parity of the degree of forms, we have $\varepsilon \circ D = -D \circ \varepsilon$, and one can also show that $D \circ H_t = H_t \circ D$. Together with the fact that traces are invariant under cyclic permutations of the operators in a composition, we get

$$\operatorname{tr}(\varepsilon \circ D \circ D \circ H_t) = -\operatorname{tr}(D \circ \varepsilon \circ D \circ H_t) = -\operatorname{tr}(\varepsilon \circ D \circ H_t \circ D) = -\operatorname{tr}(\varepsilon \circ D \circ D \circ H_t),$$

which shows that

$$\partial_t(\operatorname{tr}(H_t|_{\Omega^{\text{even}}(M)}) - \operatorname{tr}(H_t|_{\Omega^{\text{odd}}(M)})) = 0,$$

so that the identity holds for all finite $t > 0$. Note that this requires careful justification of the steps involved (since we are working with infinite-dimensional spaces limits do not always commute, and so on). In fact, it is precisely due to these kinds of subtleties that the identity does not hold when $t = 0$.

In Euclidean space one can use the heat kernel,

$$k_t(p, q) = \frac{e^{-|p-q|^2/4t}}{(4\pi t)^{n/2}}$$

to solve the heat equation. On a general Riemannian manifold this does not work, even if $|p - q|^2$ is replaced by $d(p, q)^2$. However, it can be used to construct a good first approximation of a heat kernel, and it turns out that there exists a heat kernel $k_t \in C^\infty(\Lambda^* T^* M \boxtimes (\Lambda^* T^* M)^*)$ for the heat equation for differential forms, which satisfies

$$(H_t\omega)(p) = \int_M (k_t(p,q))(\omega(q)) \text{ vol}(q).$$

The notation $\Lambda^* T^* M \boxtimes (\Lambda^* T^* M)^*$ has not been explained yet, but in particular it means that $k_t(p,q)$ is a linear map $\Lambda^* T^*_q M \to \Lambda^* T^*_p M$, for each choice of $p, q \in M$. The heat kernel has a so-called "asymptotic expansion" of the form

$$k_t(p,q) \sim \frac{e^{-d(p,q)^2/4t}}{(4\pi t)^{n/2}}(\Phi_0(p,q) + t\Phi_1(p,q) + t^2\Phi_2(p,q) + \cdots),$$

where the terms Φ_i are also sections of the bundle $\Lambda^* T^* M \boxtimes (\Lambda^* T^* M)^*$, independent of t.

From the equation relating H_t and k_t, one can express the trace of H_t in a way analogous to the calculation of the trace of a matrix by summing the diagonal elements:

$$\operatorname{tr}(H_t) = \int_M \operatorname{tr}(k_t(p,p)) \text{ vol}(p),$$

and one can similarly express the trace of the composition with the grading operator using a pointwise grading operator:

$$\operatorname{tr}(\varepsilon \circ H_t) = \int_M \operatorname{tr}(\varepsilon \circ k_t(p,p)) \text{ vol}(p).$$

By means of the asymptotic expansion of the heat kernel, one can also derive an "asymptotic expansion" of the trace we are interested in:

$$\begin{aligned}\operatorname{tr}(\varepsilon \circ H_t) \sim \frac{1}{(4\pi t)^{n/2}}\Big(&\int_M \operatorname{tr}(\varepsilon \circ \Phi_0(p,p)) \text{ vol}(p) \\ &+ t\int_M \operatorname{tr}(\varepsilon \circ \Phi_1(p,p)) \text{ vol}(p) + t^2\int_M \operatorname{tr}(\varepsilon \circ \Phi_2(p,p)) \text{ vol}(p) + \cdots\Big).\end{aligned}$$

In our case $n = 2$ and vol $= dA$, so we get

$$\begin{aligned}\operatorname{tr}(\varepsilon \circ H_t) \sim \frac{1}{4\pi}\Big(&t^{-1}\int_M \operatorname{tr}(\varepsilon \circ \Phi_0(p,p))\, dA(p) \\ &+ \int_M \operatorname{tr}(\varepsilon \circ \Phi_1(p,p))\, dA(p) + t\int_M \operatorname{tr}(\varepsilon \circ \Phi_2(p,p))\, dA(p) + \cdots\Big).\end{aligned}$$

But we just saw that the left hand side is constant! This means that all terms except the constant term (of order t^0) must vanish, and thus

$$\operatorname{tr}(\varepsilon \circ H_t) = \frac{1}{4\pi}\int_M \operatorname{tr}(\varepsilon \circ \Phi_1(p,p))\, dA(p).$$

The terms Φ_i satisfy certain recursive differential relations, stemming from the fact that they "approximately" represent the heat flow for small t. Their values $\Phi_i(p,q)$ are in general not explicitly computable, but the diagonal values $\Phi_i(p,p)$ (which are linear endomorphisms of the space of differential forms at a point) are in principle computable in terms of curvature quantities (and their derivatives). Although these computations become cumbersome for large i, it turns out that in this case, $\mathrm{tr}(\varepsilon \circ \Phi_1(p,p))$ is the scalar curvature $\mathrm{scal}(p)$. Since the scalar curvature is twice the Gaussian curvature, it follows that $\mathrm{tr}(\varepsilon \circ H_t) = \frac{1}{2\pi} \int_M K(p)\, dA(p)$, which gives the Gauss–Bonnet formula.

Of course, there is a large number of fine details which have been glossed over in the above discussion. In this course, we will consider all of these small details, making the right definitions and proving needed results, so that we can justify the above argument. Furthermore, we will learn how these ideas generalize to other differential operators on other vector bundles.

Second meeting (May 12, 2026)

Chapter 3 and some parts of Chapter 5 were discussed.

Third meeting (May 26, 2026)

We will discuss Chapter 5.

References

[1] L. Bandara. *Boundary Value Problems and Index Theory.* Lecture notes, Universität Potsdam. 2021. URL: `https://lashi.org/toc/publications/bvpit-25012022.pdf`.

[2] C. Bär. *Spin Geometry.* Lecture notes, Universität Potsdam. 2011. URL: `https://www.math.uni-potsdam.de/fileadmin/user_upload/Prof-Geometrie/Dokumente/Lehre/Veranstaltungen/SS11/spingeo.pdf`.

[3] J. Roe. *Elliptic operators, topology and asymptotic methods.* Second. Vol. 395. Pitman Research Notes in Mathematics Series. Longman, Harlow, 1998, pp. ii+209. ISBN: 0-582-32502-1.